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Cohomology rings and formality properties of nilpotent groups

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ABSTRACT

We analyze k-stage formality and relate resonance with this type of formality properties. For instance, we show that, for a finitely generated nilpotent group that is k-stage formal, the resonance varieties are trivial up to degree k. We also show that the cohomology ring, truncated up to degree $k+1$, of a finitely generated nilpotent, k-stage formal group is generated in degree 1; this criterion is necessary and sufficient for a finitely generated, 2-step nilpotent group to be k-stage formal. We compute resonance varieties for Heisenberg-type groups and deduce the degree of partial formality for this class of groups.

1. Introduction and statement of results

A path-connected space whose minimal model is isomorphic to the minimal model of its cohomology ring is called *formal*. This notion was introduced by Sullivan in his seminal work [20]. In other words, the "rational homotopy type" of the space is a formal consequence of its cohomology ring. Compact Kähler manifolds (in particular, smooth complex projective varieties) are important examples of formal spaces [5]. There is a classical weaker property, depending only on the fundamental group of the space, namely 1-*formality*, equivalent to quadratic presentability of the Malcev Lie algebra associated to the fundamental group. See e.g. [1] (Definition 3.15, Proposition 3.20).

More recently, in [11, Definition 2.2], Fernández and Muñoz introduced an interesting notion of partial formality (called k-formality by the authors), for $1 \leq k \leq \infty$, and analyzed it in detail. Since their notion is different from the classical one for $k = 1$ (as we show in Example 2.6), we call that notion *k-formality in the sense of F(ernández and) M(uñoz)*.

In Lemma 2.7 of [11], the authors also consider implicitly another notion of partial formality, which we will refer to as *k-stage formality*, for $1 \leq k \leq \infty$. For $k = \infty$, both partial notions coincide with Sullivan's full formality, as follows easily from [11]. In Lemma 2.7 of [11] it is argued that, for any $1 \leq k < \infty$, the notion of k-formality in the sense of FM is equivalent to k-stage formality, but it was realized thereafter that k-formality in the sense of FM only implies k-stage formality; see the erratum to [11].

Our goal in this paper is to investigate k-stage formality. See Definition 2.1, and also (2.2) for an equivalent reformulation. This notion seems highly natural; for $k = 1$, it coincides with the classical 1-formality in the sense of [1] (as follows from (2.2)), and we hope to convince the reader that it has several other advantages over k-formality in the sense of FM. In Remark 5.4, we show with a number of examples that, for any $1 \leq k < \infty$, k-formality in the sense of FM is *strictly* stronger than k-stage formality. In particular, for $1 < k < \infty$, the second notion seems to be new. This difference is due to the fact that the test of partial formality in the sense of FM involves cohomology in infinitely many degrees, whereas our k-stage formality test is a finite one. More precisely, the latter test uses only information provided by k-minimal models, up to degree $k+1$; see Proposition 3.1(1), and Remark 3.2.

It is well known that 1-formality is the first general obstruction in the Serre problem regarding the characterization of *projective groups* (fundamental groups of smooth projective complex varieties); see Morgan [17]. A difficult particular case

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of this problem turns out to be that of nilpotent groups. A positive answer is given by Campana [2] for a certain class of 2-step nilpotent groups, the Heisenberg groups $\mathcal{H}_n$ when $n \geq 4$ (see Definition 5.1). As for the remaining Heisenberg groups, Serre noticed that $\mathcal{H}_1$ does not pass the 1-formality criterion (see Corollary 5.7 for a slightly more general result); $\mathcal{H}_2$ and $\mathcal{H}_3$ are also non-projective groups (see Carlson and Toledo [3, Corollary 4.5]). According to [3], for 3-step nilpotent groups the answer is not known.

In Corollary 5.9 we spell out some homotopy theoretic features of the smooth projective varieties constructed in [2]; this gives additional insight into the solution of the Serre problem given by Campana [2]. We deduce these homotopy theoretic features from a more general result (where partial formality of a group G is defined to be the partial formality of the classifying space $K(G, 1)$).

Theorem 1.1. *Let $k \geq 2$ and let M be a k-stage formal space such that $\pi_1(M)$ is not k-stage formal. Then, for some $2 \leq i \leq k$, the higher homotopy group $\pi_i(M)$ is non-trivial.*

The above result no longer holds when we replace k-stage formality by k-formality in the sense of FM; see Remark 2.8.

Some of the results presented in Section 3, such as that involving the passage from partial to full formality (Proposition 3.4) and Proposition 3.1(2) are inspired by similar results in the 1-connected case due to Papadima [19].

For a finitely generated nilpotent group, we develop obstructions to k-stage formality involving either the generators of the *truncated cohomology ring* or certain *resonance varieties* (see Definition 4.1) associated to the cohomology ring. Given a non-negatively graded algebra H^*, we will denote by $H^{\leq k+1}$ the quotient algebra $H^*/H^{>k+1}$ obtained by truncation up to degree $k + 1$.

Theorem 1.2. *Let G be a finitely generated nilpotent group.*

(1) *If G is k-stage formal, then $H^{\leq k+1}(G)$ is generated as an algebra by $H^1(G)$.*
(2) *For a 2-step nilpotent group, the converse holds as well.*

For $k = 1$, the first part of Theorem 1.2 follows from [1], Lemma 3.17, and the second part follows from [3], Corollary 0.2.1. As explained in Remark 4.6, even for $k = 1$, the second part may fail to hold when G is not 2-step nilpotent.

Over a field $\Bbbk$ of characteristic zero, another obstruction to partial formality can be phrased in terms of the resonance varieties $\mathcal{R}^i_1(G) \subseteq H^1(G, \Bbbk)$. We explain this in the following result generalizing Lemma 2.4 in [3]; that lemma corresponds to the case $s = 1$ below.

Theorem 1.3. *The resonance varieties of a finitely generated, nilpotent s-stage formal group G are trivial up to degree s, that is, $\mathcal{R}^i_1(G) \subseteq \{0\}$ for $i \leq s$.*

It is well known that the fundamental group of the complement of a complex hyperplane arrangement is 1-formal. It turns out that the nilpotency test from Theorem 1.3, via resonance, is faithful for this class of groups (see Example 4.4).

In Section 5 we explore the formality properties of Heisenberg-type groups from a double perspective—generators of the cohomology ring and resonance varieties.

2. Partial minimal models and formality properties

In this section, we review the definition of k-stage formality and we prove Theorem 1.1 from the Introduction, in a slightly stronger form. We begin by recalling Sullivan's theory of minimal models from [20] and his celebrated notion of full formality. Other basic references in rational homotopy theory used in this paper are [5,10,12] and [17].

Let (A^*, d_A) be a differential graded-commutative algebra (D.G.A.) over a field $\Bbbk$ of characteristic zero such that $H^0(A^*, d_A)$ is the ground field. A minimal model for A^* is a minimal D.G.A. $(\mathcal{M}, d_{\mathcal{M}})$ such that there exists a morphism of D.G.A.'s $\rho : \mathcal{M} \longrightarrow A^*$ inducing a cohomology isomorphism. Up to isomorphism, there is a unique minimal model, $\mathcal{M} = \mathcal{M}(A)$. Let K be a space having the homotopy type of a connected simplicial complex. The minimal model of K, denoted by $\mathcal{M}(K)$, is the minimal model associated to the D.G.A. of piecewise linear forms $\Omega^*(K)$.

A D.G.A. A^* as above is *formal* if there exists a D.G.A. map

$$(\mathcal{M}(A), d_{\mathcal{M}}) \longrightarrow (H^*(A), d = 0)$$

inducing a cohomology isomorphism. A space K is *formal* if the minimal model of K is a formal D.G.A, i.e., $\mathcal{M}(K) = \mathcal{M}(H^*(K), d = 0)$.

A minimal algebra $\mathcal{M}$ generated by elements of degree $\leq k$ is a *k-minimal model* of a D.G.A. (A^*, d_A) if there exists a D.G.A. map $\rho : \mathcal{M} \longrightarrow A$ such that ρ induces in cohomology isomorphisms up to degree k and a monomorphism in degree $k + 1$ [17, Definition 5.3]. Up to isomorphism, there is a unique k-minimal model, $\mathcal{M} = \mathcal{M}_k(A)$. For a space K, set $\mathcal{M}_k(K) := \mathcal{M}_k(\Omega^*(K))$. If $\mathcal{M} = (\wedge V, d)$ is a minimal algebra, then $\mathcal{M}_k(\mathcal{M}) = (\wedge V^{\leq k}, d)$, as follows from [17, p. 165]. Hence, $\mathcal{M}_k(K) = (\wedge V^{\leq k}, d)$, if $\mathcal{M}(K) = (\wedge V, d)$.

We will investigate the following natural notion of partial formality.

Definition 2.1. A D.G.A. (A^*, d_A) is called *k-stage formal* if there is a sequence of D.G.A. morphisms connecting (A^*, d_A) to $(H^*(A), 0)$, not necessarily going all in the same direction, such that each of them induces an isomorphism in cohomology up to degree k and a monomorphism in degree $k + 1$. A space K is called *k-stage formal* if the D.G.A. $\Omega^*(K)$ is k-stage formal.

Lemma 2.2. *A D.G.A. (A^*, d_A) is k-stage formal if and only if there exists a D.G.A. morphism $(\mathcal{M}_k(A), d) \xrightarrow{\phi} (H^*(A), 0)$ which induces isomorphisms in cohomology up to degree k and a monomorphism in degree $k+1$.*

Proof. Assume there is a sequence of morphisms connecting (A^*, d_A) to $(H^*(A), 0)$, as in Definition 2.1. Notice that if there exists a morphism between two D.G. algebras inducing an isomorphism in cohomology up to degree k and a monomorphism in degree $k+1$, then the two algebras have isomorphic k-minimal models, by uniqueness. Hence, $\mathcal{M}_k(A^*, d_A) = \mathcal{M}_k(H^*(A), 0)$, and the existence of ϕ follows from the definition of $\mathcal{M}_k$.

Conversely, given ϕ we obtain a chain of D.G.A. maps, as in Definition 2.1,

$$(A^*, d_A) \xleftarrow{\rho} (\mathcal{M}_k(A), d) \xrightarrow{\phi} (H^*(A), 0), \tag{2.1}$$

where ρ is the map from the definition of the k-minimal model of A. $\square$

We infer that a space K with minimal model $(\wedge V, d)$ is k-stage formal if and only if there is a morphism of D.G.A's,

$$\phi : (\wedge V^{\leq k}, d) \longrightarrow (H^*(K), d = 0) \tag{2.2}$$

such that the map $\phi^* : H^*(\wedge V^{\leq k}, d) \longrightarrow H^*(K)$ induced on cohomology is an isomorphism up to degree k and a monomorphism in degree $k+1$. In other words, K is k-stage formal if and only if

$$\mathcal{M}_k(K) = \mathcal{M}_k(H^*(K), 0). \tag{2.3}$$

In particular, K is 1-stage formal if and only if K is 1-formal in the classical sense [1, Definition 3.15]. Since $(\wedge V^{\leq 0}, d) = (\Bbbk \cdot 1, d = 0)$, (2.2) indicates that every path-connected space K should be considered 0-stage formal.

A group G is formal when the associated Eilenberg–MacLane space $K(G, 1)$ is formal; it is k-stage formal when the associated Eilenberg–MacLane space $K(G, 1)$ is k-stage formal.

To simplify notation, we denote $\mathcal{M}(K(G, 1))$ by $\mathcal{M}(G)$ and $\mathcal{M}_k(K(G, 1))$ by $\mathcal{M}_k(G)$.

Remark 2.3. By comparing (2.2) and Lemma 2.7 in [11, erratum], we conclude that k-formality in the sense of FM implies k-stage formality, for any $1 \leq k \leq \infty$. Note that, for any $1 \leq r < s \leq \infty$, s-stage formality implies r-stage formality; this follows directly from Definition 2.1. A partial converse will be proved in Proposition 3.4 below. Likewise, for any $1 \leq k \leq \infty$, $(k+1)$-formality in the sense of FM implies k-formality in the sense of FM, (see [11, p. 150]).

Recall that a continuous map between connected CW-complexes, $f : X \longrightarrow Y$, is a *k-homotopy equivalence* if it induces isomorphisms on homotopy groups up to degree $k-1$ and a surjection in degree k. A k-homotopy equivalence is a *homology k-equivalence*, that is, a map which induces isomorphisms on cohomology groups up to degree $k-1$ and a monomorphism in degree k; see [22]. If f is a homology k-equivalence, the definition of a partial minimal model implies that

$$\mathcal{M}_{k-1}(X) = \mathcal{M}_{k-1}(Y) \quad \text{and} \quad \mathcal{M}_{k-1}(H^*(X), 0) = \mathcal{M}_{k-1}(H^*(Y), 0). \tag{2.4}$$

Corollary 2.4. *Assume $f : X \to Y$ is a homology k-equivalence. Then X is $(k-1)$-stage formal if and only if Y is $(k-1)$-stage formal. In particular, X is 1-stage formal if and only if $\pi_1(X)$ is 1-stage formal.*

Proof. The first statement follows easily using the partial formality test (2.3) and (2.4). For the second claim, consider the classifying map $f : X \to K(\pi_1(X), 1)$, which is a homology 2-equivalence. $\square$

Remark 2.5. This result is similar to [11, Theorem 5.2(i)], where the authors prove only the implication "Y is $(k-1)$-formal in the sense of FM $\Rightarrow X$ is $(k-1)$-formal in the sense of FM".

We point out that k-formality in the sense of FM is actually *strictly* stronger than k-stage formality, for any $1 \leq k < \infty$. We shall see this in the next example, for $k = 1$, and in Remark 5.4, for arbitrary k.

Example 2.6. The Heisenberg group $\mathcal{H}_4$ is 1-formal (that is, 1-stage formal), but not 1-formal in the sense of FM. Indeed, according to Campana [2], $\mathcal{H}_4$ can be realized as the fundamental group of a smooth projective variety, known to be a formal space. Using Remark 2.3 and Corollary 2.4 we deduce that $\mathcal{H}_4$ is 1-stage formal. By a classical result of Malcev [15], on finitely generated, torsion-free nilpotent groups, $\mathcal{H}_4$ can also be realized as the fundamental group of a compact nilmanifold M. Lemma 2.6 from [11] implies that $\mathcal{H}_4$ cannot be 1-formal in the sense of FM unless M is a torus, which would contradict the fact that $\mathcal{H}_4$ is non-abelian.

Theorem 1.1 follows from the Theorem 2.7 below.

Theorem 2.7. *Assume either M is a k-stage formal space such that $\pi_1(M)$ is not k-stage formal or M is not k-stage formal and $\pi_1(M)$ is k-stage formal, where $k \geq 2$. Then $\pi_i(M) \neq 0$, for some $2 \leq i \leq k$.*

Proof. Suppose $\pi_i(M) = 0$ for $2 \leq i \leq k$ and set $\pi_1(M) := H$. Consider the classifying map $f : M \longrightarrow K(H, 1)$. This map is a $(k+1)$-homotopy equivalence. Then M is k-stage formal if and only if H is k-stage formal, by Corollary 2.4, which contradicts our hypothesis. $\square$

Remark 2.8. With k-formality in the sense of FM instead of k-stage formality, Theorem 1.1 no longer holds. Let M be the smooth complex projective variety with fundamental group $\mathcal{H}_n$ constructed by Campana [2] for $n \geq 4$. We know that M is formal, hence k-formal in the sense of FM, for any k, but $\pi_1(M)$ is not even 1-formal in the sense of FM, by [11, Lemma 2.6]. Were Theorem 1.1 true for $k = 2$ and k-formality in the sense of FM instead of k-stage formality, it would imply that $\pi_2(M) \neq 0$. One can deduce that $\pi_2(M) = 0$ from the construction of M (see [2], or [3, Section 5]), when $n \geq 6$.

3. Partial formality and generators of the cohomology ring

In this section we prove Theorem 1.2, and we show that a k-stage formal CW-complex of dimension at most $k+1$ is formal. Both results are based on a characterization of k-stage formality inspired from [5] and [11].

We will need several basic properties of the bigraded minimal model of a connected, graded-commutative algebra H^*, extracted from [12]. As an algebra, $\mathcal{B} = \wedge Z$, where Z is bigraded by $Z = \oplus_{i\geq 0,\ p\geq 1} Z_i^p$. The differential d is compatible with the bigrading, that is, d has degree $+1$ with respect to upper degrees and is of degree -1 with respect to lower degrees. Moreover,

$$H_+(\mathcal{B}, d) = 0. \tag{3.1}$$

The k-minimal model of $\mathcal{B}$ will be denoted by ${}_k\mathcal{B} := (\wedge Z^{\leq k}, d)$.

For a D.G.A. $(\wedge V, d)$, set $C^* = \mathrm{Ker}(d|_{V^*})$. We recall that a space M is a *rational* $K(\pi, 1)$ if $\mathcal{M}(M) = (\wedge V, d)$ has the property that $V^* = V^1$. This property holds for $M = K(G, 1)$ when G is a finitely generated nilpotent group and, moreover $\dim_{\Bbbk} V < \infty$; see [20].

Proposition 3.1. (1) *A D.G.A. (A^*, d_A) is k-stage formal ($1 \leq k \leq \infty$) if and only if it has a k-minimal model $\mathcal{M}_k = (\wedge V^{\leq k}, d)$ with a decomposition $V^{\leq k} = C^{\leq k} \oplus N^{\leq k}$ such that $(N \cdot \mathcal{M}_k \cap \mathrm{Ker}\ d)^{\leq k+1} \subseteq d\mathcal{M}_k$, where $N = \oplus_{1\leq i\leq k} N^i$.*

(2) *Suppose that $\mathcal{M}_k = (\wedge V^{\leq k}, d)$ has the property spelled out under heading (1) above. Then the map $\phi : \wedge C^{\leq k} \longrightarrow H^*(\mathcal{M}_k)$ of graded algebras which associates to an element in $\wedge C^{\leq k}$ the cohomology class of that element is surjective up to degree $k+1$.*

(3) *If M is a k-stage formal space and a rational $K(\pi, 1)$, then the cohomology algebra of M, truncated up to degree $k+1$, is generated by $H^1(M)$.*

Proof. (1) If A^* is k-stage formal, then Lemma 2.2 implies that $\mathcal{M}_k(A) = {}_k\mathcal{B}$, where $\mathcal{B}$ is the bigraded model of $H^*(A)$. The required decomposition of $Z^{\leq k}$ is as follows: $C^q = Z_0^q$ and $N^q = \oplus_{i>0} Z_i^q$, for $q \leq k$. Let $x \in N \cdot_k \mathcal{B} \cap \mathrm{Ker}\ d$ be homogeneous of upper degree $q \leq k+1$. The lower degree of each component of x is strictly positive. By (3.1), $x = d(z)$, for some $z \in \mathcal{B}^{q-1}$. Actually $z \in_k \mathcal{B}^{q-1}$, since $q \leq k+1$.

To prove the converse claim, define a morphism of graded algebras, $\rho : \mathcal{M}_k \longrightarrow H^*(\mathcal{M}_k)$, by $c \mapsto [c]$ and $n \mapsto 0$, for $c \in C^{\leq k}$, $n \in N^{\leq k}$. We begin by showing that ρ is a D.G.A. map, that is, $\rho(d(v)) = 0$, for any $v \in V^{\leq k}$. We can write $d(v) = \overline{v} + \overline{c}, \overline{v} \in N \cdot \mathcal{M}_k, \overline{c} \in \wedge C^{\leq k}$. Then $\overline{v} \in (N \cdot \mathcal{M}_k \cap \mathrm{Ker}\ d)^{\leq k+1}$, so $\overline{v}$ is a boundary in $\mathcal{M}_k$, which implies $\rho(d(v)) = [\overline{c}] = 0$.

Let us prove now the injectivity of the map $H^q(\rho)$, for $q \leq k+1$. Take $\alpha \in \mathcal{M}_k^q$ such that $d(\alpha) = 0$ and $H^q(\rho)([\alpha]) = 0$. Write $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \wedge C^{\leq k}$ and $\alpha_2 \in N \cdot \mathcal{M}_k$. It follows that $d(\alpha_2) = 0$, hence $\alpha_2 = d(z), z \in \mathcal{M}_k$, so $[\alpha] = [\alpha_1]$. Clearly, $H^q(\rho)([\alpha]) = [\alpha_1]$ whence $[\alpha] = 0$, since $[\alpha_1] = 0$.

This also establishes the surjectivity of $H^q(\rho)$, for $q \leq k+1$. Indeed, given $\alpha \in \mathcal{M}_k^q$ such that $d(\alpha) = 0$, $H^q(\rho)([\alpha_1]) = [\alpha_1] = [\alpha]$.

To end the proof, via Lemma 2.2, consider the composition of ρ with the map induced in cohomology by the map from the definition of the k-minimal model.

(2) The claim follows from the proof of the surjectivity of $H^q(\rho)$, for $q \leq k+1$.

(3) By k-stage formality, M has a k-minimal model $\mathcal{M}_k$ which, in turn, has the property spelled out in Proposition 3.1(1).

Since M is a rational $K(\pi, 1)$, we conclude that $V^{\leq k} = V^1$, $C^{\leq k} = C^1$ and $\mathcal{M}_k = \mathcal{M}(M)$. Our claim follows by considering the map $\phi : \wedge C^1 \longrightarrow H^*(M)$ from Part (2). □

Remark 3.2. The inclusion $(N \cdot \mathcal{M}_k \cap \mathrm{Ker}\ d)^{\leq k+1} \subseteq d\mathcal{M}_k$ is equivalent to the inclusion $(N \cdot \mathcal{M}_k \cap \mathrm{Ker}\ d)^{\leq k+1} \subseteq d\mathcal{M}$, since $\mathcal{M}^{\leq k} = \mathcal{M}_k^{\leq k}$. We chose the first formulation to emphasize the fact that one may use only k-minimal models to define k-stage formality.

Using the second formulation, a direct comparison of Proposition 3.1(1) with the definition of k-formality in the sense of FM [11, Definition 2.2] shows once more that the notion of k-stage formality is less restrictive than k-formality in the sense of FM, for all k. The point is that Fernández and Muñoz require exactness of the elements from $N \cdot \mathcal{M}_k \cap \mathrm{Ker}\ d$ in *all* degrees, while we demand this only up to degree $k+1$. Note also that the k-formality in the sense of FM test involves the *full* minimal model.

Let $\psi : \wedge C^{\leq k} \to H^*(\mathcal{M}(A))$ be the composition of ϕ with the map induced in cohomology by the inclusion $\mathcal{M}_k \subset \mathcal{M}(A)$. The authors of [11] show that ψ is onto, up to degree k, when the D.G.A. (A^*, d_A) is k-formal in the sense of FM. Our Proposition 3.1(2) may be viewed as an extension of their result to the k-stage formal case.

Proof of Theorem 1.2. (1) Since $K(G, 1)$ is a rational $K(\pi, 1)$, the claim follows from Proposition 3.1(3).

(2) The minimal model of a 2-step nilpotent group G is of the form

$$\mathcal{M}(G) = (\wedge(x_1, \ldots, x_m) \otimes \wedge(y_1, \ldots, y_n), d),$$

where $\deg(x_i) = \deg(y_j) = 1$, C^1 is the $\Bbbk$-span of $\{x_1, \ldots, x_m\}$, denoted $\langle x_1, \ldots, x_m\rangle$, and $d(y_j) \in \wedge^2(x_1, \ldots, x_m)$ for all j.

Define a graded algebra map, $\phi : (\mathcal{M}, d) \longrightarrow (H^*(\mathcal{M}), 0)$, by $x_i \mapsto [x_i]$ and $y_j \mapsto 0$. It is easy to check that ϕ is a D.G.A. morphism and $H^1(\phi)$ is the identity. Since the algebra $H^{\le k+1}(G)$ is supposed to be generated in degree 1, $H^{\le k+1}(\phi)$ must be the identity. This proves the k-stage formality of G, by Lemma 2.2. □

We will need the following lemma.

Lemma 3.3. *Let (A^*, d_A) be a D.G.A.. Any k-minimal model $\mathcal{M}_k = (\wedge V^{\le k}, d)$ can be extended to a $(k+1)$-minimal model $\mathcal{M}_{k+1} = (\wedge V^{\le k+1}, d)$ such that for any $v \in V^{k+1}$, $d(v) \in d(\mathcal{M}_k)$ if and only if $d(v) = 0$. Moreover, $V^{k+1} = \oplus_{i\ge 0} V_i^{k+1}$ and $d(v) = 0$ if and only if $v \in V_0^{k+1}$.*

Proof. This relies on a standard inductive construction. Start with a k-minimal model map, $\phi_k : (\wedge(V^{\le k}), d) \to (A^*, d_A)$. At step 0, extend it to $\phi_{k0} : (\wedge(V^{\le k} \oplus V_0^{k+1}), d) \to (A^*, d_A)$, where $V_0^{k+1} = \operatorname{coker} H^{k+1}(\phi_k)$ and $d|_{V_0^{k+1}} = 0$, to achieve surjectivity in cohomology in degree $k+1$. At step $i+1$, extend

$$\phi_{ki} : (\wedge(V^{\le k} \oplus V_{\le i}^{k+1}), d) \to (A^*, d_A)$$

by killing $\operatorname{Ker} H^{k+2}(\phi_{ki})$. By construction, the transgression map, $[d] : V_{i+1}^{k+1} \to H^{k+2}(\wedge(V^{\le k} \oplus V_{\le i}^{k+1}), d)$, is injective. Finally, $V^{k+1} = \oplus_{i\ge 0} V_i^{k+1}$.

We need to show that, given $v = \sum_{i=0}^n v_i \in V^{k+1}$, where $v_i \in V_i^{k+1}$, $d(v) \in d(\mathcal{M}_k)$ implies $v \in V_0^{k+1}$. If $d(v) = d(\alpha)$, for some $\alpha \in \mathcal{M}_k$, then

$$d(v_n) = d\left(\alpha - \sum_{i=0}^{n-1} v_i\right) \in d(\wedge(V^{\le k} \oplus V_{<n}^{k+1})).$$

If $n > 0$, this forces $v_n = 0$, by the injectivity of the transgression. Hence, $v = v_0$. □

Proposition 3.4. *A k-stage formal space M with $H^{\ge k+2}(M) = 0$ is formal. A k-stage formal CW-complex of dimension at most $k+1$ is formal.*

Proof. Clearly, the second assertion follows from the first. By Proposition 3.1(1), case $k = \infty$, we have to find a minimal model $\mathcal{M} = (\wedge V, d)$ that admits a decomposition $V^i = C^i \oplus N^i$ for any i, such that $C^i = \operatorname{Ker}(d|_{V^i})$ and any closed element in $N \cdot \mathcal{M}$ is exact, where $N = \oplus_{i\ge 1} N^i$. The k-stage formality guarantees the existence of a k-minimal model $\mathcal{M}_k = (\wedge V^{\le k}, d)$ with a decomposition $V^i = C^i \oplus N^i$ for $i \le k$, such that any closed element of degree at most $k+1$ in $N^{\le k} \cdot \mathcal{M}_k$ is exact in $\mathcal{M}_k$. Extend $\mathcal{M}_k$ to $\mathcal{M}_{k+1}$ as in Lemma 3.3, and then extend $\mathcal{M}_{k+1}$ to a minimal model $\mathcal{M}$.

We first extend the decomposition of the algebra generators, from $\mathcal{M}_k$ to $\mathcal{M}$. We know from Lemma 3.3 that $C^{k+1} = V_0^{k+1}$. We set $N^{k+1} = \oplus_{i>0} V_i^{k+1}$, and $N^{\ge k+2} = V^{\ge k+2}$. We need to show that $C^{\ge k+2} = 0$. Indeed, if $\alpha \in V^i$, $i \ge k+2$ and $d(\alpha) = 0$, then $\alpha = d(z)$, since $H^{\ge k+2}(\mathcal{M}) = 0$. Minimality implies that exact elements of $\mathcal{M}$ must be decomposable, hence $\alpha = 0$, as asserted.

It remains to check that every closed homogeneous element $\alpha \in (N \cdot \mathcal{M})^i$ is exact in $\mathcal{M}$. If $i \ge k+2$, this is clear, since $H^{\ge k+2}(\mathcal{M}) = 0$. If $i \le k$, then $\alpha \in N^{\le k} \cdot \mathcal{M}_k$. Hence, α must be exact in $\mathcal{M}$, by Proposition 3.1(1). Finally, if $i = k+1$, then $\alpha = \alpha_1 + \alpha_2$, with $\alpha_1 \in (N^{\le k} \cdot \mathcal{M}_k)^{k+1}$ and $\alpha_2 \in N^{k+1}$. Then $d(\alpha) = 0$ implies $d(\alpha_2) = d(-\alpha_1)$, hence $\alpha_2 = 0$, by Lemma 3.3. The exactness of $\alpha = \alpha_1$ follows again by Proposition 3.1(1). □

We remark that Lemma 2.10 from [11] is a consequence of Proposition 3.4.

Corollary 3.5. *Any space arising as the complement of a complex plane projective curve is formal.*

Proof. One knows that complements of plane projective curves are 1-formal spaces, having the homotopy type of a CW-complex of dimension at most 2; see [14] and [6] respectively for details. The corollary follows from Proposition 3.4. □

The same result was proved in [4], using a different approach.

4. Partial formality and resonance

This section is devoted to the proof of Theorem 1.3. We also examine the sharpness of this result. We start by recalling the definition of resonance varieties.

They were defined by Falk [9] in relation to complex hyperplane arrangements, and were then intensively studied in this context. For a more general definition and further exploration of that concept, see [7] and the references therein.

Definition 4.1. Let H^* be a connected graded-commutative $\Bbbk$-algebra. The resonance variety $\mathcal{R}_k^q(H^*)$ is the subset of those elements $w \in H^1$ such that $\dim_{\Bbbk} H^q(H^*, \mu_w) \ge k$, where μ_w is the differential given by left-multiplication by w in H^*; this is a homogeneous algebraic subvariety of the affine space H^1, when H^* is of finite type as a graded vector space. We denote $\mathcal{R}_k^q(H^*)$ by $\mathcal{R}_k^q(M)$ when M is a path-connected space and $H^* = H^*(M, \Bbbk)$; for $M = K(G, 1)$ we use the notation $\mathcal{R}_k^q(G)$.

Since $H^0 = \Bbbk \cdot 1$, it follows that $\mathcal{R}_1^0(H^*) = \{0\}$. We will relate resonance to partial formality, by using the lemma below as a basic ingredient.

Lemma 4.2. *Let H^* be a connected graded-commutative algebra and denote by $\mathcal{B}$ the bigraded minimal model of H^* constructed in [12]. If $\mathcal{R}_1^q(H^*) \not\subseteq \{0\}$, then the vector space $\mathcal{B}^q$ has infinite dimension.*

Proof. Property (3.1) implies that all elements of H^* have representatives belonging to $\mathcal{B}_0$. With this remark, the hypothesis of the lemma means that there exist $0 \neq \omega \in Z_0^1$ and $\eta \in \mathcal{B}_0^q$, $q > 0$, such that $[\eta\omega] = 0$ in $H^{q+1}(\mathcal{B})$, and $[\eta] \notin H^{q-1}(\mathcal{B}) \cdot [\omega]$. We will use this to construct inductively a sequence of elements $(\alpha_n)_{n \geq 0}$ such that $\alpha_n \in \mathcal{B}_n^q \setminus \{0\}$ and $d(\alpha_{n+1}) = \alpha_n \omega$, for $n \geq 0$, which will finish the proof.

Set $\alpha_0 = \eta$. Clearly, $\alpha_0 \neq 0$. Since $[\eta\omega] = 0$, $\eta\omega = d(\alpha_1)$, with $\alpha_1 \in \mathcal{B}_1^q$. If $\alpha_1 = 0$, then $\eta\omega = 0$ in the free graded-commutative algebra $\mathcal{B}$. Consequently, $\eta = \overline{\eta}\omega$ with $\overline{\eta} \in \mathcal{B}_0^{q-1}$, which implies $[\eta] = [\overline{\eta}][\omega]$, a contradiction.

The passage from $(\alpha_{\leq n})$ to α_{n+1}, for $n \geq 1$, goes as follows. Since $d(\alpha_n \omega) = d(\alpha_n)\omega = \alpha_{n-1}\omega^2 = 0$, $\alpha_n \omega = d(\alpha_{n+1})$, with $\alpha_{n+1} \in \mathcal{B}_{n+1}^q$, by (3.1). We are left with checking that $\alpha_{n+1} \neq 0$. Assuming the contrary, we infer that $\alpha_n \omega = 0$. This equality leads to a contradiction.

Indeed, it implies that $\alpha_n = \zeta_n \omega$, with $\zeta_n \in \mathcal{B}_n^{q-1}$. Differentiating the last equality we obtain $\alpha_{n-1}\omega = d(\zeta_n)\omega$, hence $\alpha_{n-1} - d(\zeta_n) = \zeta_{n-1}\omega$, with $\zeta_{n-1} \in \mathcal{B}_{n-1}^{q-1}$. Differentiating again we obtain $\alpha_{n-2}\omega = d(\zeta_{n-1})\omega$, which implies $\alpha_{n-2} - d(\zeta_{n-1}) = \zeta_{n-2}\omega$, with $\zeta_{n-2} \in \mathcal{B}_{n-2}^{q-1}$. Eventually we obtain the equality $\eta - d(\zeta_1) = \zeta_0 \omega$, with $\zeta_1 \in \mathcal{B}_1^{q-1}$ and $\zeta_0 \in \mathcal{B}_0^{q-1}$, which implies $[\eta] = [\zeta_0][\omega]$. This is the desired contradiction. $\square$

Proof of Theorem 1.3. Let us denote by $\mathcal{B}$ the bigraded minimal model of the cohomology algebra $H^*(G)$, and by ${}_s\mathcal{B}$ the s-minimal model of $H^*(G)$. According to the previous lemma, it is enough to show that $\mathcal{B}^q$ has finite dimension, for $q \leq s$. The s-stage formality of G implies that the s-minimal model of G coincides with the s-minimal model of $H^*(G)$, that is, $\mathcal{M}_s(G) = {}_s\mathcal{B}$. Since G is finitely generated and nilpotent, $\mathcal{M}(G) = \mathcal{M}_1(G) = \mathcal{M}_s(G)$ is a finite dimensional vector space. Clearly, $\mathcal{B}^q =_s \mathcal{B}^q$, for $q \leq s$. Our proof is complete. $\square$

Given a graded-commutative algebra H^*, let K be the kernel of the multiplication map $\mu : H^1 \wedge H^1 \longrightarrow H^2$, called in [3] the characteristic subspace of H^*. It is proved in [3, Lemma 2.4] that the characteristic subspace of $H^*(G, \Bbbk)$ contains no non-trivial decomposables, when G is a finitely generated, nilpotent, 1-formal group. This result may be recovered from our Theorem 1.3, via the lemma below.

Lemma 4.3. *The subspace K contains no non-trivial decomposables if and only if*

$$\{\omega \in K | \omega^2 = 0 \in \wedge^4 H^1\} \subseteq \{0\}. \tag{4.1}$$

Both properties are equivalent to $\mathcal{R}_1^1(H^*) \subseteq \{0\}$.

Proof. Given $0 \neq \omega \in H^1 \wedge H^1$, write $\omega = \sum_{i=1}^m x_i \wedge y_i$, in canonical form. Assume $\omega^2 = 0$. Since clearly $\omega^m \neq 0$, we infer that $\omega = x_1 \wedge y_1$ is decomposable. This shows that the first property in the lemma implies (4.1). Conversely, a decomposable element $\omega = \alpha_1 \wedge \alpha_2$ satisfies $\omega^2 = 0$.

By Definition 4.1, $\mathcal{R}_1^1(H^*) \not\subseteq \{0\}$ if and only if there are $0 \neq \alpha \in H^1$ and $\beta \in H^1$ such that $\alpha \wedge \beta \neq 0$ and $\mu(\alpha \wedge \beta) = 0$. This shows that the first and the third property in the lemma are equivalent. $\square$

Example 4.4. Here we use [18] as a basic reference for arrangement theory. Let $G_{\mathcal{A}} = \pi_1(M_{\mathcal{A}})$ be the fundamental group of the complement of a central complex hyperplane arrangement $\mathcal{A} \subset \mathbb{C}^n$, $n \geq 3$. It is known that $G_{\mathcal{A}}$ is finitely generated and 1-stage formal, since $M_{\mathcal{A}}$ is a formal space. We claim that the following properties are equivalent:

(1) The hyperplanes of $\mathcal{A}$ are in general position in codimension 2.
(2) The group $G_{\mathcal{A}}$ is abelian.
(3) The group $G_{\mathcal{A}}$ is nilpotent.
(4) $\dim_{\mathbb{Q}} \operatorname{gr}(G_{\mathcal{A}}) \otimes \mathbb{Q} < \infty$.
(5) $\mathcal{R}_1^1(G_{\mathcal{A}}) \subseteq \{0\}$.
(6) $\mathcal{V}_1^1(G_{\mathcal{A}}) \subseteq \{1\}$.

(Here $\operatorname{gr}(G) \otimes \mathbb{Q}$ is the rational graded Lie algebra associated to the lower central series of G, and $\mathcal{V}_1^1(G)$ denotes the first characteristic variety of G in degree one; see e.g. [7] for the definitions.)

Indeed, the implication (1) $\Rightarrow$ (2) follows from Hattori's Theorem in [13], and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. For (4) $\Rightarrow$ (1), we refer to [8, Proposition 2.12]. The implication (3) $\Rightarrow$ (5) is given by our Theorem 1.3 and (5) $\Rightarrow$ (1) is implicit in the proof of Proposition 2.12 from [8]. For (3) $\Rightarrow$ (6), we use [16, Theorem 1.1]. Finally, (6) $\Rightarrow$ (5) holds for all finitely generated 1-formal groups, due to the local analytic isomorphism $\exp : (\mathcal{R}_1^1, 0) \xrightarrow{\cong} (\mathcal{V}_1^1, 1)$; see [7, Theorem A].

This shows in particular that the nilpotency condition in Theorem 1.3 is necessary. Indeed, if $\mathcal{A}$ is a braid arrangement, then $G_{\mathcal{A}}$ is finitely generated and 1-stage formal, but the hyperplanes of $\mathcal{A}$ are not in general position in codimension 2 (see [18]), hence $\mathcal{R}_1^1(G_{\mathcal{A}}) \not\subseteq \{0\}$.

In the next example, we will see that the triviality of the resonance varieties, up to a fixed degree, is not a sufficient condition for partial formality, even when the corresponding truncated cohomology ring is generated in degree one.

Example 4.5. Let $\mathcal{B} = (\wedge(x_1, x_2, y_1, y_2, z, \omega_1, \omega_2, \alpha), d)$ be the 1-stage formal minimal D.G.A. generated in degree 1 from [3], Example 2.8, where $d(x_i) = d(y_i) = d(z) = 0$, $d(\omega_1) = x_1y_1 + x_2z$, $d(\omega_2) = x_2y_2 + x_1z$, $d(\alpha) = x_1\omega_1 + x_2\omega_2$. The bigrading of $\mathcal{B} = \wedge Z$ defined by $Z_0 = \langle x_i, y_i, z\rangle$, $Z_1 = \langle \omega_1, \omega_2\rangle$, $Z_2 = \langle\alpha\rangle$ is compatible with d, in the sense explained at the beginning of Section 3. Moreover, it is easy to check that $H^{\leq 2}_{+}(\mathcal{B}) = 0$.

Given an arbitrary element $p \in \wedge^2 Z_0$, define an upper degree 1 derivation D of the algebra $\wedge Z$ as follows: $D|_{Z_0,Z_1} = d|_{Z_0,Z_1}$ and $D(\alpha) = d(\alpha) + p$. A direct computation shows that $D^2 = 0$. Denote by $\mathcal{M} = (\wedge Z, D)$ the corresponding minimal D.G.A. Both $\mathcal{B}$ and $\mathcal{M}$ are minimal D.G.A.'s, finitely generated in degree one. As explained in [20], they may be realized by finitely presentable (3-step) nilpotent groups: $\mathcal{B} = \mathcal{M}(G_{\mathcal{B}})$ and $\mathcal{M} = \mathcal{M}(G_p)$.

We claim that $H^{\leq 2}(\mathcal{M}) \cong H^{\leq 2}(\mathcal{B})$, as algebras. By applying Theorems 1.2 and 1.3 to $G_{\mathcal{B}}$, we infer that the algebra $H^{\leq 2}(G_p) \cong H^{\leq 2}(G_{\mathcal{B}})$ is generated in degree 1 and $\mathcal{R}^1_1(G_p) \cong \mathcal{R}^1_1(G_{\mathcal{B}}) \subseteq \{0\}$.

To verify our claim, we start by noting that $H^1(\mathcal{M}) = H^1(\mathcal{B}) = Z_0$. Next, we show that $\mathrm{Ker} d|_{\mathcal{B}^2} = \mathrm{Ker} D|_{\mathcal{M}^2}$, and compute this vector space explicitly. For $\mathcal{B}$, the result is $\wedge^2 Z_0 + \langle d(\alpha)\rangle$. This follows from $H^{\leq 2}_{+}(\mathcal{B}) = 0$, for lower degree reasons. Any element of $\wedge^2 Z$ is of the form $w = \overline{w} + \xi\alpha$, where $\overline{w} \in \wedge^2(Z_0 \oplus Z_1)$ and $\xi \in Z_0 \oplus Z_1$. If $0 = D(w) = d(\overline{w}) + d(\xi)\alpha - \xi d(\alpha) - \xi p$, then $d(\xi) = 0$, hence $\xi \in Z_0$. Write $\overline{w} = \sum_{i=0}^{2} \overline{w}_i$, with $\overline{w}_i \in (\wedge^2 Z_{\leq 1})_i$. Equating lower degree 1 components in $D(w) = 0$, we find that $\xi = 0$, using the description of d-closed elements in upper degree 2. We conclude that $\mathrm{Ker} d|_{\mathcal{B}^2} = \mathrm{Ker} D|_{\mathcal{M}^2}$. Since clearly $\wedge^2 Z_0 + \langle D(\alpha)\rangle = \wedge^2 Z_0 + \langle d(\alpha)\rangle$, the algebra $H^{\leq 2}(\mathcal{M})$ is generated by $H^1(\mathcal{M})$. Moreover, $\dim H^2(\mathcal{M}) = \dim H^2(\mathcal{B})$, since $\mathrm{im} d|_{\mathcal{B}^1} \cong \mathrm{im} D|_{\mathcal{M}^1}$.

To obtain a graded algebra isomorphism, $H^{\leq 2}(\mathcal{M}) \cong H^{\leq 2}(\mathcal{B})$, we consider the graded algebra $\mathcal{C}^* := \frac{\wedge^* Z_0}{\wedge^* Z_0 \cdot dZ_1}$. Notice that $H^{\leq 2}(\mathcal{B}) \cong \mathcal{C}^{\leq 2}$, as algebras, since $H^{\leq 2}_{+}(\mathcal{B}) = 0$. Define a graded algebra morphism $\psi : \mathcal{C}^* \to H^*(\mathcal{M})$ by $\psi(z) = [z]$, for $z \in Z_0$. It is clear that ψ^1 is a linear isomorphism, and ψ^2 is a surjection between vector spaces of the same dimension, according to the above computations. Hence, the algebras $H^{\leq 2}(\mathcal{M})$ and $H^{\leq 2}(\mathcal{B})$ are isomorphic, as asserted.

Denote by G the group corresponding to the choice $p = y_1y_2$. Note that $\mathcal{M}_1(H^*(G), 0) = \mathcal{B}$, since $H^{\leq 2}(\mathcal{M}) \cong H^{\leq 2}(\mathcal{B})$ and $\mathcal{B}$ is 1-formal. We claim that G is not 1-stage formal. Assuming the contrary, we must have a D.G.A. isomorphism $\phi : \mathcal{B} \longrightarrow \mathcal{M}$, according to (2.3).

Moreover, we can choose ϕ such that $\phi|_{Z_0} = \mathrm{id}$. Indeed, let $\tilde{h} : \mathcal{B} \xrightarrow{\sim} \mathcal{B}$ be the 1-minimal model of the graded algebra automorphism induced by ϕ, $h : H^{\leq 2}(\mathcal{B}) \xrightarrow{\sim} H^{\leq 2}(\mathcal{M}) \cong H^{\leq 2}(\mathcal{B})$. Replacing an arbitrary ϕ by $\phi \circ \tilde{h}^{-1}$, we obtain the desired property.

Checking the equality $D\phi = \phi d$ on ω_i, we find that $\phi(\omega_i) - \omega_i \in Z_0$, for $i = 1, 2$. Therefore, ϕ leaves the subspace $Z_0 + Z_1$ invariant. Since $\phi : Z \to Z$ is an isomorphism, necessarily $\phi(\alpha) = a_1x_1 + a_2x_2 + b_1y_1 + b_2y_2 + cz + d_1\omega_1 + d_2\omega_2 + e\alpha$, with $e \neq 0$. Since $\phi|_{Z_0} = \mathrm{id}$, $\phi(d(\alpha)) \equiv 0$, modulo the ideal generated by x_1 and x_2. Clearly, $D(\phi(\alpha)) \equiv ey_1y_2$, modulo this ideal, which contradicts the fact that ϕ is a D.G.A. morphism.

Remark 4.6. Let G be a finitely presentable group, with 1-minimal model $\mathcal{M}$. In Lemma 3.17, implication $(i) \Rightarrow (ii)$, the authors of [1] note that the algebra $H^{\leq 2}(\mathcal{M})$ is generated in degree 1, if G is 1-formal. This result can be recovered from our Proposition 3.1(1)–(2), case $k = 1$.

On the other hand, the converse of this implication does not hold, contrary to the claim from [1, Lemma 3.17]. Indeed, the finitely presentable, 3-step nilpotent group G constructed in Example 4.5 has 1-minimal model $\mathcal{M}$ with the property that the algebra $H^{\leq 2}(\mathcal{M})$ is generated in degree 1, yet G is not 1-formal.

5. Heisenberg-type groups

According to Theorems 1.2 and 1.3, applied to an arbitrary finitely generated, 2-step nilpotent group G, $\mathcal{R}^{\leq q}_1(G) \subseteq \{0\}$, as soon as the ring $H^{\leq q+1}(G)$ is generated in degree 1. As we point out in Example 5.8, the converse implication may fail to hold. For a Heisenberg-type group G, it turns out that the above two obstructions to q-stage formality are equivalent, in a certain range. Moreover, our techniques enable us to determine the degree of partial formality, for such a group G. See Corollary 5.7. We begin by examining the following well-known family of finitely generated torsion-free 2-step nilpotent groups.

Definition 5.1. The *Heisenberg group* $\mathcal{H}_n$ is given by the central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H}_n \longrightarrow \mathbb{Z}^{2n} \longrightarrow 0, \tag{5.1}$$

corresponding to the cohomology class $\omega \in H^2(\mathbb{Z}^{2n}, \mathbb{Z}) = \wedge^2_{\mathbb{Z}}(x_1, y_1, \ldots, x_n, y_n)$, where $\omega = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$.

It is natural to enlarge the class of Heisenberg groups, as follows. Let G be a finitely generated, 2-step nilpotent group defined by a central extension of the form

$$0 \longrightarrow B \longrightarrow G \longrightarrow A \longrightarrow 0, \tag{5.2}$$

where B is an abelian group of rank 1, and A is an abelian group of finite rank m. The minimal model of G is then of the form $\mathcal{M}(G) = \wedge(t_1, \ldots, t_m) \otimes \wedge(z)$, with differential given by $d(t_i) = 0$, $\forall i$ and $d(z) := \omega \in \wedge^2(t_1, \ldots, t_m)$.

Definition 5.2. The group G is called *of Heisenberg type* if $\omega \neq 0$.

We may assume ω has the canonical form $\omega = x_1 y_1 + \cdots + x_n y_n$, where $2n = \mathrm{rk}(\omega)$; consequently:

$$\mathcal{M}(G) = \mathcal{M}(\mathcal{H}_n) \otimes (\wedge(t_{2n+1}, \ldots, t_m), d = 0). \tag{5.3}$$

Lemma 5.3. *The cohomology of the Heisenberg group $\mathcal{H}_n$ is given by*

$$H^q(\mathcal{H}_n) \cong \frac{\wedge^q(x_i, y_i)}{\omega \wedge^{q-2}(x_i, y_i)} \oplus \{\eta z \mid \eta\omega = 0, \eta \in \wedge^{q-1}(x_i, y_i)\}, \quad \forall q. \tag{5.4}$$

The second summand is trivial, for $q \leq n$, and non-trivial, for $q = n + 1$.

Proof. It is clear that $H^1(\mathcal{H}_n) = \wedge^1(x_i, y_i)$. Let us compute $H^q(\mathcal{H}_n)$, for $2 \leq q$. Any q-form $\xi \in \wedge^q(x_i, y_i, z)$ may be written $\xi = \eta_1 + \eta_2 z$, where $\eta_1 \in \wedge^q(x_i, y_i)$ and $\eta_2 \in \wedge^{q-1}(x_i, y_i)$. Hence, ξ is a cocycle if and only if $\eta_2\omega = 0$. When $q \leq n$, the last equality implies $\eta_2 = 0$, by the hard Lefschetz theorem, see [21]. Clearly, the q-coboundaries coincide with the elements of the form $\eta\omega$, with $\eta \in \wedge^{q-2}(x_i, y_i)$. Consequently, $H^q(\mathcal{H}_n)$ has the asserted form. Clearly, $\eta = y_1 \cdots y_n$ creates a non-trivial contribution of the second summand, in degree $n + 1$. $\square$

Remark 5.4. The above lemma shows that $\mathcal{H}_n$ is $(n-1)$-stage formal (use Theorem 1.2, Part (2)), but not n-stage formal (as follows from Theorem 1.2, Part (1)). Moreover, for any $1 \leq n < \infty$, the Heisenberg group $\mathcal{H}_{n+1}$ is not n-formal in the sense of FM. Indeed, n-formality in the sense of FM would imply 1-formality in the sense of FM (see Remark 2.3), hence abelianity of $\mathcal{H}_{n+1}$, by [11, Lemma 2.6].

Next, we consider the resonance varieties of Heisenberg groups. For $n = 1$, the result below shows in particular that partial formality is needed in Theorem 1.3.

Proposition 5.5. *For $q \leq n - 1$, $\mathcal{R}_1^q(\mathcal{H}_n) = \{0\}$, while $\mathcal{R}_1^n(\mathcal{H}_n) = \Bbbk^{2n}$.*

Proof. We know $H^*(\mathcal{H}_n)$ (see Lemma 5.3). We first compute the cohomology of the complex $H^*(\mathcal{H}_n)$, with respect to the differential given by left-multiplication with the class of an element $0 \neq \xi \in \wedge^1(x_i, y_i)$.

We may assume $\xi = x_1$, by a linear change of coordinates. The n-class $[y_1 \ldots y_n]$ satisfies $[\xi][y_1 \ldots y_n] = 0$, since $x_1 y_1 \ldots y_n = \omega y_2 \ldots y_n$. Assuming $[y_1 \ldots y_n] \in [\xi]H^{n-1}(\mathcal{H}_n)$, we infer that $y_1 \ldots y_n = x_1\eta + \omega\beta$, with $\eta \in \wedge^{n-1}(x_i, y_i)$ and $\beta \in \wedge^{n-2}(x_i, y_i)$; reducing this equality modulo the ideal generated by $x_1, \ldots, x_n$, we obtain a contradiction. Therefore, $[\xi] \in \mathcal{R}_1^n(\mathcal{H}_n)$. This proves that $\wedge^1(x_i, y_i) \setminus \{0\} \subseteq \mathcal{R}_1^n(\mathcal{H}_n)$. Since $\mathcal{R}_1^n(\mathcal{H}_n)$ is Zariski closed in $\wedge^1(x_i, y_i)$, we infer that $\mathcal{R}_1^n(\mathcal{H}_n) = \Bbbk^{2n}$, as asserted.

Theorem 1.3 implies that $\mathcal{R}_1^q(\mathcal{H}_n) \subseteq \{0\}$, for $q < n$. It remains to check that $0 \in \mathcal{R}_1^q(\mathcal{H}_n)$, if $q < n$. Assuming the contrary, it follows from Lemma 5.3 that $H^n(\mathcal{H}_n) = 0$. But this implies $\mathcal{R}_1^n(\mathcal{H}_n) = \emptyset$, contradicting the first computation. $\square$

One may use the following result to obtain information on the resonance varieties associated to Heisenberg-type groups:

Proposition 5.6. *Let A^*, B^* be connected graded-commutative algebras. Then*

$$\mathcal{R}_1^q(A^* \otimes B^*) = \bigcup_{m+n=q} \mathcal{R}_1^m(A^*) \times \mathcal{R}_1^n(B^*). \tag{5.5}$$

Proof. Set $C^* = A^* \otimes B^*$. If $\xi = \xi_A + \xi_B \in C^1 = A^1 \oplus B^1$ is an arbitrary degree 1 element, then the multiplication by ξ on C^* is given by:

$$\mu_\xi(a \otimes b) = \mu_{\xi_A}(a) \otimes b + (-1)^{|a|} a \otimes \mu_{\xi_B}(b), \quad a \in A^*, \ b \in B^*. \tag{5.6}$$

By Künneth, (5.5) follows from (5.6). $\square$

Corollary 5.7. *A Heisenberg-type group G with $\mathrm{rk}(\omega) = 2m$ is $(m-1)$-stage formal, but not m-stage formal. Moreover, $\mathcal{R}_1^q(G) = \{0\}$, for $q \leq m - 1$, and $\mathcal{R}_1^m(G) = \Bbbk^{2m}$.*

Proof. The claims on partial formality follow from Theorem 1.2, Part (2), by using Lemma 5.3 and (5.3) to describe the cohomology ring of G up to degree $m + 1$.

By (5.3) and Proposition 5.6, $\mathcal{R}_1^k(G)$ equals

$$\mathcal{R}_1^k(H^*(\mathcal{H}_m) \otimes \wedge^*(t_{2m+1}, \ldots, t_n)) = \bigcup_{p+q=k} \mathcal{R}_1^p(\mathcal{H}_m) \times \mathcal{R}_1^q(\wedge^*(t_{2m+1}, \ldots, t_n)).$$

Since the resonance for the exterior algebra is trivial, the claims on resonance varieties follow from Proposition 5.5. $\square$

In conclusion, the resonance test provided by Theorem 1.3 detects precisely the stage of partial formality, for the family of Heisenberg-type groups. In general, the triviality of the resonance varieties and the generation in degree 1 of the truncated cohomology ring are independent properties, as we will see below.

Example 5.8. Let G be a finitely generated, 2-step nilpotent group with minimal model $\mathcal{N} = \wedge(x_1, x_2, y_1, y_2, z) \otimes \wedge(\omega_1, \omega_2)$, generated in degree 1 and having differential given by $dz = dx_i = dy_i = 0$, $i = 1, 2$ and $d\omega_1 = x_1y_1 + x_2z$, $d\omega_2 = x_2y_2 + x_1z$; see Example 4.5. Then $\mathcal{R}_1^{\leq 1}(G) \subseteq \{0\}$, but the ring $H^{\leq 2}(G)$ is not generated in degree 1.

First we compute the cohomology of $\mathcal{N}$ in low degrees. It is clear that $H^1(\mathcal{N}) = \wedge^1(x_i, y_i, z)$ and $H^2(\mathcal{N}) = H_0^2(\mathcal{N}) \oplus H_1^2(\mathcal{N}) \oplus H_2^2(\mathcal{N})$, where $H_0^2(\mathcal{N}) = \frac{\wedge^2(x_i, y_i, z)}{\langle d\omega_1, d\omega_2 \rangle}$. A direct computation shows that $H_1^2(\mathcal{N}) = \langle [x_1\omega_1 + x_2\omega_2] \rangle$, and $H_2^2(\mathcal{N}) = 0$. In particular, $H^{\leq 2}(G)$ is not generated in degree 1, since $H_1^2(\mathcal{N}) \neq 0$.

To compute $\mathcal{R}_1^1(G)$, take one-cycles, $0 \neq \xi \in \wedge^1(x_i, y_i, z)$ and $\eta \in \wedge^1(x_i, y_i, z)$, such that $[\eta\xi] = 0$ in cohomology. This implies that $\eta\xi = ad\omega_1 + bd\omega_2$ for some $a, b \in \Bbbk$. It follows that $0 = (\eta\xi)^2 = 2a^2x_1y_1x_2z + 2abx_1y_1x_2y_2 + 2b^2x_2y_2x_1z$, so $a = b = 0$. Consequently, $\eta\xi = 0$ in $\wedge^2(x_1, x_2, y_1, y_2, z)$. Since $\xi \neq 0$, $\eta \in \langle \xi \rangle$. Therefore, $\mathcal{R}_1^{\leq 1}(G) \subseteq \{0\}$.

Corollary 5.9. *A Heisenberg-type group G with $\mathrm{rk}(\omega) = 2m$ cannot be realized as the fundamental group of a smooth projective complex variety M with $\pi_{\leq m}(\tilde{M}) = 0$, where $\tilde{M}$ is the universal covering of M.*

Proof. Assume $G = \pi_1(M)$, with M a smooth projective complex variety. By the main result of [5], M is a formal space, hence m-stage formal, while G is not m-stage formal, by Corollary 5.7. This forces $m \geq 2$, by Corollary 2.4. Consequently, $\pi_i(\tilde{M}) \cong \pi_i(M) \neq 0$, for some $2 \leq i \leq m$; see Theorem 1.1. $\square$

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